Model for the Nucleon and Baryon Regge Trajectories

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A model for pion-nucleon scattering is considered which may permit a unified dynamical explanation of the low-energy properties of the nucleon Regge trajectory and the higher $T=\frac{1}{2}$ resonances on that trajectory. The basis of the model is a field-theoretic graph which contains a vector meson and a nucleon in the intermediate state. The lowest order N/D method is used to construct a unitary scattering amplitude which contains a Regge trajectory. The trajectory passes through $J=\frac{1}{2}$ when the total energy is equal to the nucleon mass and has a reasonable slope. When generalized to an SU_3 invariant theory of baryon, pseudoscalar meson, and vector meson octets, the model predicts a value $\alpha=0.427$ for the coefficient of F-type Yukawa coupling, and suggests that a unitary singlet Y_0^* resonance of spin-parity $D_{3/2}$ exists. In an Appendix the relation between the N/D and Fredholm methods is discussed and the important role played by the factorability of the Born amplitudes is pointed out.

I. INTRODUCTION

THE analytic behavior of partial-wave scattering amplitudes in the angular momentum variable has been of considerable interest since the pioneering work of Regge[1] in potential scattering was performed. The hypothesis that elementary particles and resonances are associated with moving poles in the angular momentum plane has been proposed by Chew and Frautschi[2] and by Blankenbecler and Goldberger. The existence of certain groups of these particles and resonances is correlated by this hypothesis in the sense that they may lie on the same Regge trajectory. The trajectories in turn govern the high-energy behavior of cross sections and resolve certain theoretical difficulties concerning particles of high spin.[3]

The desired meromorphic property of scattering amplitudes in the angular momentum plane has been proved and the leading Regge trajectory computed to lowest order in the coupling constant in three cases: potential scattering,[4] ladder graphs for two scalar mesons scattering by scalar meson exchange,[5] and the ladder graphs for the scattering of a scalar meson and a spin-$\frac{1}{2}$ Fermion by scalar meson exchange.[6] In these cases the trajectory is computed by power-series expansion of the denominator function of a Fredholm integral equation,[7] and to lowest order the only contribution comes from the residue of the fixed angular momentum pole of the Born term. The interacting elementary particle can lie on the trajectory generated only for special choices of the coupling constant.

Gell-Mann, Goldberger, Low, and Zachariasen[8] suggest on the basis of a study of perturbation theory that a spin-$\frac{1}{2}$ Fermion does lie on a Regge trajectory determined by its interaction with a vector meson. In their method, the Born terms for the scattering of a vector meson and Fermion are studied, and the effect of higher order terms in the perturbation series is simulated by using a simple N/D method with an appropriately chosen subtraction point. Again only the fixed pole in the Born term contributes to the trajectory in lowest order.

In this paper a method similar to that of GGLZ is used to construct Fermion Regge trajectories. In particular a model of the nucleon Regge trajectory is found which reflects some of the phenomenological features of pion-nucleon scattering. This model is then generalized to an SU_3 invariant theory and the resulting baryon trajectory is determined. A field-theoretic box graph containing a vector meson and nucleon in the intermediate state is the basis for our model. The lowest order N/D method is used to generate the Regge trajectory. A subtraction in the denominator function is made to insure that the nucleon or baryon lie on the trajectory, and the model then predicts the slope.

In the $T=\frac{1}{2}$ channel of pion-nucleon scattering, there are the $P_{1/2}$ nucleon at 939 MeV, a $D_{3/2}$ $N_{1/2}^*$ resonance at 1512 MeV, and a $F_{5/2}$ $N_{1/2}^*$ resonance at 1688 MeV. These masses and quantum numbers are taken from a compilation of Rosenfeld.[9] In addition, there has recently been reported an $N_{1/2}^*$ resonance[10] at 2190 MeV, which has been assigned the quantum numbers $G_{7/2}$ on the basis of an empirical rule.[11] These states are

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* National Science Foundation Cooperative Graduate Fellow.
[1] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947 (1960).
[2] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1962). Mention of the work of Blankenbecler and Goldberger is made in the second reference.
[3] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).
[4] Lecture by B. W. Lee in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963).
[5] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).
[6] D. Amati, A. Stanghellini, and K. Wilson, Nuovo Cimento **28**, 639 (1963). The method used by these authors does not provide an explicit proof of the meromorphy of the partial-wave amplitudes. The method of Ref. 5 can be adapted to this case, and the present author has proven the desired meromorphy properties by this method.
[7] M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958).
[8] M. Gell-Mann, M. L. Goldberger, F. E. Low, and F. Zachariasen, Phys. Letters **4**, 265 (1963); M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962).
[9] A. H. Rosenfeld, Lawrence Radiation Laboratory (unpublished).
[10] A. N. Diddens, E. W. Jenkins, T. F. Kycia, and K. F. Riley, Phys. Rev. Letters **10**, 262 (1963).
[11] T. F. Kycia and K. F. Riley, Phys. Rev. Letters **10**, 266 (1963).

plotted on a Chew-Frautschi[2] type diagram in Fig. 1. It is seen that the two Regge trajectories on which these states lie are quite close together. This indicates[12] that exchange forces which normally split the trajectories of even and odd partial waves[3] are rather unimportant in this case, and we may be successful by considering a model which contains no exchange force. It should be noted that the GGLZ model is pure exchange force, and if our observations are correct, such a force cannot explain the observed pattern of resonant states.

We choose the box graph for pion-nucleon scattering in which the $T=1$ vector ρ meson appears in the intermediate state [Fig. 2(a)]. This graph has been used previously in models explaining the $D_{3/2}$ and $F_{5/2}$ resonances by Cook and Lee[13] and by Ball, Frazer, and Nauenberg.[14] It is our aim to show that it also furnishes a reasonable model of the nucleon Regge trajectory in the vicinity of the nucleon mass, so that a unified dynamical model of the whole nucleon Regge trajectory might indeed be possible. This graph contains no exchange force, and it provides appropriate ratios for the strengths of forces in the representations of both SU_2 and SU_3. The fixed pole in the box graph amplitude, which would occur at $l=-1$ for scalar particles, is displaced by the spin of the intermediate state to $J=\frac{1}{2}$, the value appropriate to a Regge-pole Fermion.

In Sec. II, the determination of the nucleon trajectory is discussed, and in Sec. III its generalization to an SU_3 invariant theory is given and the properties of the resulting trajectories discussed. In Appendix A, a discussion is given of the relation of the N/D and Fredholm methods in field-theoretic models and of the role played by the factorability of the Born amplitudes.[8] The method used to calculate the partial-wave amplitude for the box graph is outlined in Appendix B.

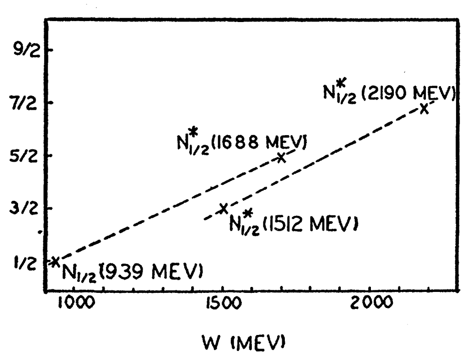

FIG. 1. Chew-Frautschi type plot of the $T=\frac{1}{2}$ pion-nucleon states. Angular momentum is plotted on the ordinate, and the energy or mass of the state is given by the abscissa.

[12] This observation and its significance were first noted by R. F. Sawyer.

[13] L. F. Cook, Jr., and B. W. Lee, Phys. Rev. **127**, 283 (1962).

[14] J. S. Ball, W. R. Frazer, and M. Nauenberg, Phys. Rev. **128**, 478 (1962).

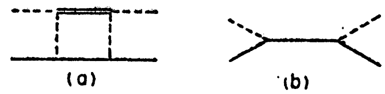

FIG. 2. Feynman graphs referred to in the text. (a) Box graph. (b) Single Fermion graph. In the work of Sec. II, the dashed line is the π meson, the solid line the nucleon, and the double line the ρ meson. In Sec. III, the dashed line is a member of the pseudoscalar meson octet, the solid line is a member of the baryon octet, and the double line a member of the vector meson octet.

II. NUCLEON TRAJECTORY

In the neighborhood of $l=1$ the f_{l-} $(J=l-\frac{1}{2})$ amplitude[15] for the box graph of Fig. 2(a) is given by

$$\begin{pmatrix} B_{l-}{}^{1/2}(w) \\ B_{l-}{}^{3/2}(w) \end{pmatrix} = \frac{(w-m_0)^2-\mu^2}{w^2} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \frac{G^2F^2}{8\pi} h^+(w) \frac{1-\delta_{l1}}{l-1}. \quad (1)$$

Here w is the total energy $w=s^{1/2}$, and m_0, m, and μ are the masses of nucleon, ρ and π, respectively. G^2 is the square of the ordinary pion-nucleon coupling constant $G^2=15$, and F^2 is the square of the $\rho\rightarrow 2\pi$ coupling constant $F^2=2-2.5$. The upper coefficient refers to the $T=\frac{1}{2}$ state and the lower coefficient to the $T=\frac{3}{2}$ state. The noncontinuable δ_{l1} term comes from the partial-wave projection and serves to make the physical partial-wave amplitude finite at $l=1$, that is, in our notation $1-\delta_{l1}$ means that the singularity is not present in the $l=1$ amplitude.

The function $h(w)$ is given by

$$h^{\pm}(w)=w^{-1}\{\ln(m/m_0)+[(w\pm m_0)^2-m^2] \times|(w^2-(m+m_0)^2)(w^2-(m-m_0)^2)|^{-1/2}I(w)\}, \quad (2)$$

and

$$\begin{aligned} I(w) &= (\pi/2)+\tan^{-1}[w^2-(m^2+m_0^2)] \times|(w^2-(m+m_0)^2)(w^2-(m-m_0)^2)|^{-1/2} \\ &\qquad \text{for} \quad (m_0-m)^2<w^2<(m_0+m)^2 \\ &= \ln[w^2-(m^2+m_0^2)-|(w^2-(m+m_0)^2) \times(w^2-(m-m_0)^2)|^{1/2}][2mm_0]^{-1} \\ &\qquad \text{for} \quad w^2<(m_0-m)^2 \quad \text{or} \quad w^2>(m_0+m)^2. \quad (3) \end{aligned}$$

We have written the kinematic factors[15] to the left in (1), and it is seen that the amplitude contains a pole at $w=0$ as a remnant of left-hand singularities and a cusp at the ρ production threshold.

Our method is to unitarize the contribution of this graph by the lowest order N/D method in the w plane. The denominator function contains the right-hand two-particle unitarity cuts, and because of the MacDowell symmetry[16]

$$f_{l-}(w)=-f_{l-1+}(-w), \quad (4)$$

These cuts occur for both positive and negative w. In

[15] We normalize our amplitudes so that $f=q^{-1}e^{i\delta}\sin\delta$. See S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[16] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

the lowest order expressions we have

$$N(w)=B_{l-}(w)\,,$$

$$D(w)=1-\frac{1}{\pi}\left[\int_{m_0+\mu}^{\infty}dw'\frac{q(w')}{w'-w}B_{l-}(w')+\int_{-\infty}^{-(m_0+\mu)}dw'\frac{q(w')}{w'-w}B_{l-}(w')\right].\quad(5)$$

Here $q(w')$ is the momentum in the center-of-mass system

$$q^2(w)=[(w^2-(m_0-\mu)^2)(w^2-(m_0+\mu)^2)]/(4w^2).\quad(6)$$

If the Bethe-Salpeter equation with our graph as kernel were of Fredholm type, our method would be identical to the determinantal method of Baker.[7] But since the Fredholm traces diverge, a subtraction is necessary in the denominator function, and we take it at the nucleon mass. After multiplying numerator and denominator by $l-1$ and making an obvious change of integration variable in D, we obtain for our case

$$f_{l-}(w)=\bar{N}(w)\bar{D}^{-1}(w)\,,\quad(7)$$

where

$$\begin{pmatrix}\bar{N}^{1/2}(w)\\ \bar{N}^{3/2}(w)\end{pmatrix}=\frac{(w-m_0)^2-\mu^2}{w^2}\frac{G^2F^2}{8\pi}\begin{pmatrix}4\\1\end{pmatrix}h^+(w)(1-\delta_{l1})\,,$$

$$\begin{pmatrix}\bar{D}^{1/2}(w)\\ \bar{D}^{3/2}(w)\end{pmatrix}=l-1-\frac{(w-m_0)}{8\pi^2}G^2F^2\begin{pmatrix}4\\1\end{pmatrix}\times\int_{m_0+\mu}^{\infty}dw'\frac{q(w')}{w'^2}\left\{\frac{(w'-m_0)^2-\mu^2}{(w'-m_0)(w'-w)}h^+(w')-\frac{(w'+m_0)^2-\mu^2}{(w'+m_0)(w'+w)}h^-(w')\right\}.\quad(8)$$

This expression has the Regge form

$$f_{l-}(w)=\beta(w)/[l-\alpha(w)],\quad(9)$$

and it is seen that the trajectory passes through $l=1$ at $w=m_0$ because of the subtraction. Hence, the nucleon lies on the trajectory. According to the mechanism suggested by GGLZ, in the $T=\frac{1}{2}$ state the contribution of the noncontinuable term will be cancelled at the nucleon pole by the single nucleon graph of Fig. 2(b), while in the $T=\frac{3}{2}$ state the noncontinuable term remains so that (7) does not imply the existence of a particle at $J=\frac{1}{2}$ in this state.

The graph of Fig. 2(b) is pure p-wave $T=\frac{1}{2}$ and has a pole at the nucleon mass. Near the pole the amplitude of this graph is identical to that of the complete $f_{l-}^{1/2}$ amplitude and is given by

$$f_{l-}^{1/2}(w)=\frac{(w-m_0)^2-\mu^2}{4w^2}\frac{-3G^2}{(w-m_0)}.\quad(10)$$

Comparison of the residue of (10) with the residue of the analytic part of (7) at $l=1$ gives the following expression for the coupling constant of the theory:

$$G^2=4\pi h^+(m_0)\left[3\int_{m_0+\mu}^{\infty}dw'\frac{q(w')}{w'^2}\left\{\frac{(w'-m_0)^2-\mu^2}{(w'-m_0)^2}h^+(w')-\frac{(w'+m_0)^2-\mu^2}{(w'+m_0)^2}h^-(w')\right\}\right]^{-1}.\quad(11)$$

The Regge-pole condition thus determines the coupling constant in this model, and it differs from GGLZ in this respect.

We have no reason to believe in the numerical value of this coupling constant, since it depends on an integration over the high-energy region where our unitarity approximation is not valid. Indeed, the value of the integral depends on a cancellation between a positive contribution from the region $m_0+\mu<w<m_0+m$ and a negative contribution of nearly equal magnitude from the high-energy region. Because of this, the value of G^2 obtained is much too large ($G^2\approx100$) and quite difficult to calculate accurately by numerical methods. We note that Frye and Warnock[17] have recently given a formulation of N/D in which the complete unitarity condition is satisfied. In this formulation the integrand of D is modified above the production threshold by the inelasticity parameter. This parameter, not well determined in our model, depends on G^2, so that the coupling constant would be determined by a self-consistent equation of the form

$$G^2=f(G^2)\quad(12)$$

instead of by Eq. (11). Computations based on an artificial form of the inelasticity parameter indicate that much smaller values of G^2 can be obtained in this way.

In the expression for the slope,

$$\alpha'(m_0)=\left.\frac{d\alpha(w)}{dw}\right|_{w=m_0},\quad(13)$$

of the trajectory in (8), we use Eq. (11). The result then depends only on the numerator function evaluated at $w=m_0$, and is given by

$$\begin{pmatrix}\alpha'^{1/2}(m_0)\\ \alpha'^{3/2}(m_0)\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}\frac{F^2}{6\pi}h^+(m_0)=\begin{pmatrix}0.19\\0.05\end{pmatrix}\frac{1}{\mu}.\quad(14)$$

The average slope of the Regge trajectory which passes through the nucleon at 939 MeV and the $F_{5/2}$ resonance at 1688 MeV is

$$\alpha'^{1/2}|_{\mathrm{av}}=0.37\frac{1}{\mu},\quad(15)$$

so that our model indeed gives a reasonable slope in the $T=\frac{1}{2}$ state.

[17] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

The motivation for the subtraction in D is not as clear in the $T=\frac{3}{2}$ amplitude, and there is no cancellation of the noncontinuable term. There is, consequently, less reason to take the trajectory appearing in this amplitude seriously. If we do take it seriously, we see that the slope is down by a factor of 4, and it is plausible that in this model the $T=\frac{3}{2}$ trajectory never gets high enough to make a reasonance. Such a result is in agreement with experiment because the $T=\frac{3}{2}$ $P_{3/2}$ resonance does not lie on this trajectory for positive w. It is also well known that the force responsible for this resonance has a different dynamical origin.[18]

III. SU_3 INVARIANT CALCULATION

The model has been extended to the calculation of amplitudes generated by the box graph in the SU_3 invariant theory of baryon, pseudoscalar meson, and vector meson octets. We have taken the particle states, Clebsch-Gordan coefficients, and Yukawa Lagrangian from de Swart.[19] The trilinear Lagrangian for the vector meson, pseudoscalar, pseudoscalar interaction is

$$\begin{aligned}\mathcal{L}=(4\pi)^{1/2}F\{&-i\boldsymbol{\varrho}^\mu\cdot\boldsymbol{\pi}\times i\partial_\mu\boldsymbol{\pi}+\tfrac{1}{2}\boldsymbol{\varrho}^\mu\cdot(K^\dagger\boldsymbol{\tau}i\partial_\mu K-i\partial_\mu K^\dagger\boldsymbol{\tau}K)\\&+(\tfrac{1}{2}\sqrt{3})\omega^\mu(K^\dagger i\partial_\mu K-i\partial_\mu K^\dagger K)\\&+\tfrac{1}{2}(\sqrt{3})[M^{\dagger\mu}(Ki\partial_\mu\eta-i\partial_\mu K\eta)+\text{H.c.}]\\&+\tfrac{1}{2}[M^{\dagger\mu}(\boldsymbol{\tau}Ki\partial_\mu\cdot\boldsymbol{\pi}-\boldsymbol{\tau}i\partial_\mu K\cdot\boldsymbol{\pi})+\text{H.c.}]\}\,.\end{aligned}\tag{16}$$

The usual particle-field notation is used here; ω being a $T=0$, $S=0$ vector meson, and M being a $T=\frac{1}{2}$, $S=1$ vector K^* resonance.

Because of the well-known formula

$$\mathbf{8}\times\mathbf{8}=\mathbf{1}+\mathbf{8}+\mathbf{8}'+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{27},\tag{17}$$

meson baryon scattering is described by 7 invariant amplitudes. Four of these refer to scattering in the **1**, **10**, $\overline{\mathbf{10}}$, and **27** representations. Because two equivalent 8-fold representations occur in the reduction formula (17), transitions between states of the two representations are allowed. Scattering in the octet segment is then described by a 2×2 symmetric matrix, and this contains three independent invariant amplitudes.

The dynamical part of the box graph contribution is the same as before. The Born amplitudes are again given by (1), with the isospin coefficients replaced by a matrix A which gives the strengths of the force in the various irreducible representations of (17) as a function of α, the coefficient of F-type Yukawa coupling.[20]

$$\begin{aligned}A_{27}&=4\alpha^2\,,\\A_{10}&=A_{\overline{10}}=4(1-\alpha)^2\,,\\A_{88}&=A_{8'8'}=14\alpha^2-10\alpha+5\,,\\A_{8\leftrightarrow8'}&=-6(5)^{1/2}\alpha(1-\alpha)\,,\\A_1&=36\alpha^2.\end{aligned}\tag{18}$$

[18] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1571 (1956).
[19] J. J. de Swart, CERN Report 6488/Th. 345, 1963 (unpublished).
[20] M. Gell-Mann, California Institute of Technology Synchrotron Laboratory Report CTSL-20, 1961 (unpublished).

We unitarize these amplitudes by lowest order N/D, satisfying single-channel unitarity conditions in the **1**, **10**, $\overline{\mathbf{10}}$, and **27** representations, and a coupled channel unitarity condition in the octet segment.[21] In order for a simple Regge pole to appear in the octet amplitude, the determinant of the 2×2 coefficient matrix in the octet segment of (18) must vanish. This condition implies that $\alpha=(\sqrt{5})(3+\sqrt{5})^{-1}=0.427$, and we have taken this value in what follows. A 2×2 matrix with vanishing determinant can be written as the dyadic product of two two-dimensional vectors, so that our condition implies that the Born amplitudes can be written in factored form in the subspace of 8-fold representations.[22]

The factorability condition also has a physical interpretation in this case. If the baryon is treated as a Regge-pole by our model, it should be regarded as a vector meson bare baryon bound state. This bound state must belong to one and only one 8-fold representation of SU_3, and hence can couple only to a fixed linear combination of **8** and **8′** meson baryon states. The orthogonal linear combination couples to the Reggeized state with zero coupling strength. This means that the 2×2 matrix in question has zero as an eigenvalue and thus zero for its determinant. We note in this connection that the single baryon graph [Fig. 2(a)] has the octet amplitude

$$\begin{pmatrix}f_{88}&f_{88'}\\f_{8'8}&f_{8'8'}\end{pmatrix}=\begin{pmatrix}(5/3)(1-\alpha)^2&-(5)^{1/2}\alpha(1-\alpha)\\-(5)^{1/2}\alpha(1-\alpha)&12\alpha^2\end{pmatrix}\times\frac{-[(w-m_0)^2-\mu^2]G^2}{w^2(w-m_0)},\tag{19}$$

and the condition of zero determinant is satisfied for all values of α.

The value $\alpha=0.427$ determined by our factorability condition falls well within the range suggested by Martin and Wali[23] in their dynamical calculation of the decuplet $P_{3/2}$ resonances. It also agrees favorably with the value $\alpha=0.39$ predicted by imposing a self-consistent bootstrap condition in this calculation.[24] Our value is not out of line with the value $\alpha=0.326$ predicted by Cutkosky[25] in a self-consistent model of baryon states. In our model the value of α should remain constant along the Regge trajectory, so that we predict the value $\alpha=0.427$ for the coupling of the $D_{3/2}$ octet members to meson and baryon. This is reasonably close to the value $\alpha=0.345$ found by Glashow and Rosenfeld[26] in an empirical study.

[21] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).
[22] See Ref. 8, where a similar factorability condition in the subspace of helicity indices is required for the appearance of a Regge pole.
[23] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963).
[24] A. W. Martin and K. C. Wali (to be published).
[25] R. E. Cutkosky, Ann. Phys. (N. Y.) **23**, 415 (1963).
[26] S. L. Glashow and A. H. Rosenfeld, Phys. Rev. Letters **10**, 192 (1963).

The unitary amplitudes in our model are given by

$$f_{l-}(w)=\tilde{N}(w)\tilde{D}^{-1}(w)\,,\tag{20}$$

where

$$\tilde{N}(w)=\frac{(w-m_0)^2-\mu^2}{w^2}\frac{G^2F^2}{8\pi}A^0h^+(w)\,,$$

$$\tilde{D}(w)=l-1-\frac{(w-m_0)}{8\pi^2}G^2F^2B^0$$
$$\times\int_{m_0+\mu}^{\infty}dw'\frac{q(w')}{w'^2}\left\{\frac{(w'-m_0)^2-\mu^2}{(w'-m_0)(w'-w)}h^+(w')\right.$$
$$\left.-\frac{(w'+m_0)^2-\mu^2}{(w'+m_0)(w'+w)}h^-(w')\right\}\,,\tag{21}$$

and the coefficient matrices A^0 and B^0 are

$$A_{27}{}^0=0.77=B_{27}{}^0\,,$$
$$A_{10}{}^0=A_{\overline{10}}{}^0=1.31=B_{10}{}^0=B_{\overline{10}}{}^0\,,$$
$$A_{88}{}^0=A_{8'8'}{}^0=-A_{8'\leftrightarrow 8}{}^0=3.28=\tfrac{1}{2}B_{88}{}^0=\tfrac{1}{2}B_{8'8'}{}^0=\tfrac{1}{2}B_{8\leftrightarrow 8'}{}^0\,,\tag{22}$$
$$A_1{}^0=6.91=B_1{}^0\,.$$

Comparison of the residues of (19) and (20) at the nucleon pole again determines the coupling constant.

$$G^2=(\pi/1.09)h^+(m_0)\left[\int_{m_0+\mu}^{\infty}dw\frac{q(w)}{w^2}\left\{\frac{(w-m_0)^2-\mu^2}{(w-m_0)^2}h^+(w)-\frac{(w+m_0)^2-\mu^2}{(w+m_0)^2}h^-(w)\right\}\right]^{-1}.\tag{23}$$

The small difference of this value from (11) arises from the increased number of channels present in this case.

The slopes of the trajectories are given by

$$\alpha'(m_0)=C\frac{F^2}{8\pi}h^+(m_0)\,,\tag{24}$$

where

$$C_{27}=0.71\,,\quad C_{10}=C_{\overline{10}}=1.20\,,\quad C_{88}=C_{8'8'}=C_{8'\leftrightarrow 8}=6.02\,,\quad C_1=6.34\,.\tag{25}$$

If we again take the masses of the nucleon and ρ meson, the slope of the octet trajectory is very nearly the same as before, and this result is not very sensitive to the masses chosen.

The slopes in the 27- and 10-fold representations are small, and we may assume that these trajectories turn back before making higher angular momentum resonances. Hence, other dynamical mechanisms must be responsible for resonances in these representations if they occur experimentally. On the other hand, the slope of the unitary singlet trajectory is of the same order of magnitude as that of the octet, and we should expect both trajectories to make higher resonances. Hence, this model suggests the existence of a unitary singlet $Y_0{}^*$ resonance in the $D_{3/2}$ state. This may be the 1405-MeV $Y_0{}^*$, whose angular momentum and parity have not been definitely determined, or it may be higher at about the mass (1600 MeV) of the members of the $D_{3/2}$ octet.

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APPENDIX A

In this appendix, we discuss the relationship of the N/D and Fredholm methods in field-theoretic models of this type and point out the role played by the factorability of the Born amplitudes.

It is easily shown in all the models for which meromorphy has been proved,[4–6] that the N/D and Fredholm methods give the same denominator functions, and therefore, the same trajectories, to lowest order in the coupling constant. We illustrate this with reference to the model of Lee and Sawyer.[5] For the scattering of scalar particles of mass m by exchange of a scalar meson of mass μ, they find the Born term

$$B_l(s)=g^2(2\pi)^{-3}Q_l(1+\mu^2(2q^2)^{-1})\,,\tag{A1}$$

where Q_l is the Legendre function of second kind. This implies the normalization

$$T_l(s)=4qs^{1/2}\pi^{-2}e^{i\delta}\sin\delta\,.\tag{A2}$$

Hence, the D function would be given by

$$D_l(s)=1-\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{1}{s'-s}\frac{\pi^2}{4q's'^{1/2}}B_l(s')$$
$$=1-\frac{g^2}{8\pi^2}\int_{4m^2}^{\infty}ds'\frac{Q_l(1+\mu^2(2q'^2)^{-1})}{[s'(s'-4m^2)]^{1/2}(s'-s)}\,,\tag{A3}$$

and this agrees with Eq. (27) of Ref. 5.

In Ref. 5, it is shown that the complete Fredholm denominator $D_l(s)$ has the form

$$D_l(s)=-1-g^2f(s)(l+1)^{-1}-g^2h(l,s)\,,\tag{A4}$$

where $h(l,s)$ is regular at $l=-1$ and $f(s)$ and $h(l,s)$

approach zero as s approaches infinity. From the Regge-pole condition $D_l(s)=0$, we obtain

$$l+1=g^2f(s)(1-g^2h(l,s))^{-1}. \qquad \text{(A5)}$$

For s sufficiently large, we can make an expansion of $(1-g^2h(l,s))^{-1}$ in powers of g^2 and thus see explicitly that, to order g^2, only the singular part of $D_l(s)$ contributes to the trajectory.

The singularity of $D_l(s)$ at $l=-1$ comes from the pole term of the kernel of the Bethe-Salpeter equation, an integral equation in the two variables, q and ω, of relative momentum and relative energy. Near the pole the kernel has the form

$$\langle q\omega|K_l(s)|q'\omega'\rangle=g^2(l+1)^{-1}F^{-1}(q,\omega,s). \qquad \text{(A6)}$$

The function $F(q,\omega,s)$ is essentially the inverse product of two propagators, but for our purpose it is only necessary to notice that F is independent of q' and ω' so that the kernel (A6) is actually a dyadic in the integration space of the Bethe-Salpeter equation.

The contribution to $D_l(s)$ is given by the Fredholm determinant[7]

$$\mathrm{Det}(1-K_l)=1-\mathrm{Tr}K_l-\tfrac{1}{2}\,\mathrm{Tr}K_l^2+\tfrac{1}{2}(\mathrm{Tr}K_l)^2+\cdots, \qquad \text{(A7)}$$

and, because the kernel is a dyadic, all terms of order higher than the first in this expansion cancel out. It is this circumstance that is responsible for the simple pole in $D_l(s)$ at $l=-1$ (A4), and therefore for the simple Regge-pole behavior of the model. A kernel whose residue at the pole had finite rank n in the integration space would lead to a pole of order n in $D_l(s)$, and n Regge trajectories would be generated. For a kernel of infinite rank, $D_l(s)$ would have an essential singularity in the l plane.

In the N/D method the Born approximation appears on the mass shell only and the factorability in the integration space is not apparent. It should be noted that the factorability is responsible for the agreement between the N/D and Fredholm methods. In more complicated problems the Bethe-Salpeter equation will involve finite sums over helicity and charge channels, and the residue of the Born term would have to be factorable in the spaces of these channels as well as in the integration space in order to obtain simple Regge-pole behavior. In the N/D method for such problems the charge and helicity channels would be coupled by the unitarity condition, and factorability in these channels would be explicitly required for the generation of a simple Regge pole. It is in this manner that the factorability condition of the GGLZ model and the SU_3 model in this paper arise.[27]

In the limited context of models for which the Regge behavior can be derived from a Bethe-Salpeter equation (or Lippman-Schwinger equation in potential scattering), one can see quite directly that it is the factorability of the residue at the fixed pole in the Born term which is responsible for the desired result.

[27] See also V. Gribov and I. Pomeranchuk, Phys. Rev. Letters **8**, 343 (1962).

APPENDIX B

We outline here our method[28] of projecting out the partial-wave amplitude from the box graph amplitude in momentum space. In Fig. 3 the energy-momentum

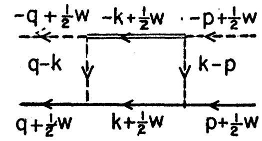

FIG. 3. Box graph with directions and momenta of particle lines indicated.

vectors of the lines in the box graph are given. In the barycentric system these vectors have the form

$$\begin{aligned} w&=(0,s)\,,\\ p&=(\mathbf{q}_i,(m_0^2-\mu^2)/2s)\,,\\ q&=(\mathbf{q}_f,(m_0^2-\mu^2)/2s)\,. \end{aligned} \qquad \text{(B1)}$$

The momentum dependence of the amplitude takes the form

$$\mathfrak{M}_{fi}=\bar{u}_f(\mathbf{q}_f)\Bigg[\int d^4k\frac{-\gamma(k+\frac{1}{2}w)+m_0}{(k+\frac{1}{2}w)^2-m_0^2+i\epsilon}\frac{1}{(q-k)^2-\mu^2+i\epsilon}$$
$$\times\frac{1}{(k-p)^2-\mu^2+i\epsilon}\frac{g^{\mu\nu}}{(-k+\frac{1}{2}w)^2-m^2+i\epsilon}$$
$$\times(\tfrac{1}{2}w+k-2p)_\mu(\tfrac{1}{2}w+k-2q)_\nu\Bigg]u_i(\mathbf{q}_i). \qquad \text{(B2)}$$

The spinor particles in the initial and final states have energy $E(q)=q_0+\frac{1}{2}(s)^{1/2}$.

We next define Dirac spinors for particles of momentum $\mathbf{k}$ and energy $E(k)=k_0+\frac{1}{2}s^{1/2}$ in terms of the fictitious mass

$$m(k)=[E^2(k)-\mathbf{k}^2]^{1/2}. \qquad \text{(B3)}$$

The positive energy spinors are given by

$$u_\alpha(\mathbf{k},m(k))=\left(\frac{E(k)+m(k)}{2m(k)}\right)^{1/2}\begin{bmatrix}\chi_\alpha\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{k}}{E(k)+m(k)}\chi_\alpha\end{bmatrix}, \qquad \text{(B4)}$$

where the σ_i are the Pauli spin matrices, and the χ_α are two-component spinors. Negative energy spinors can be defined analogously. These spinors satisfy a Dirac equation with the mass $m(k)$ and possess the usual orthogonality and completeness properties in the spin space. The γ-dependent numerator in (B1) then has

[28] This method was surgested by R. F. Sawyer.

the expansion

$$-\gamma\cdot(k+\tfrac{1}{2}w)+m_0$$
$$=\sum_\alpha\{u_\alpha(\mathbf{k},m(k))\bar{u}_\alpha(\mathbf{k},m(k))[m_0-m(k)]$$
$$-v_\alpha(\mathbf{k},m(k))\bar{v}_\alpha(\mathbf{k},m(k))[m_0+m(k)]\}. \quad \text{(B5)}$$

When we apply the final- and initial-state spinors to the left and right of (B5), a reduction to two-component spinors is effected. The first term becomes, for example,

$$\bar{u}_f(\mathbf{q}_f)\sum_\alpha u_\alpha(\mathbf{k},m(k))\bar{u}_\alpha(\mathbf{k},m(k))u_i(\mathbf{q}_i)$$
$$=\left[\frac{E(q)+m_0}{2m_0}\frac{E(k)+m(k)}{2m(k)}\right]^{1/2}$$
$$\times\left[\chi_{f},\left(1-\frac{\boldsymbol{\sigma}\cdot\mathbf{q}_f\boldsymbol{\sigma}\cdot\mathbf{k}}{[E(q)+m_0][E(k)+m(k)]}\right)\right.$$
$$\left.\times\left(1-\frac{\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{q}_i}{[E(k)+m(k)][E(q)+m_0]}\right)\chi_i\right]. \quad \text{(B6)}$$

The product of vectors in (B2) can be expanded so that the angular dependence contained in it appears explicitly:

$$g^{\mu\nu}(\tfrac{1}{2}w+k-2p)_\mu(\tfrac{1}{2}w+k-2q)_\nu$$
$$+a+b(\hat{k}\cdot\hat{q}_i+\hat{q}_f\cdot\hat{k})+c\hat{q}_f\cdot\hat{q}_i, \quad \text{(B7)}$$

and the propagators of exchanged particles can be expanded in terms of spherical harmonics and Legendre functions of the second kind.

$$\frac{1}{(k-q)^2-\mu^2+i\epsilon}=\frac{-2\pi}{|\mathbf{k}||\mathbf{q}|}\sum_{l,m}Q_l(\alpha(q,k))$$
$$\times Y_l^m(\hat{k})Y_l^{m*}(\hat{q}), \quad \text{(B8)}$$

where

$$\alpha(q,k)=[\mathbf{q}^2+\mathbf{k}^2+\mu^2-i\epsilon-(q_0-k_0)^2](2|\mathbf{q}||\mathbf{k}|).^{-1} \quad \text{(B9)}$$

The partial-wave amplitude $M_{l-}(w)$ is projected out by using the standard spherical harmonics with spin $\mathcal{Y}_m{}^{l-*}(\hat{q}_f)$ and $\mathcal{Y}_m{}^{l-}(\hat{q}_i)$ in place of the spinors χ_f and χ_i. Integration over the three solid angles $d\hat{q}_f$, $d\hat{q}_i$, and $d\hat{k}$ is tedious but straightforward, and gives the result

$$M_{l-}(w)=\frac{2\pi^2}{m_0\mathbf{q}^2}[E(q)+m_0]\int_{-\infty}^{+\infty}dk_0\int_0^\infty d|\mathbf{k}|$$
$$\times\frac{A_1+A_2+A_3}{[(k+\frac{1}{2}w)^2-m_0^2+i\epsilon][(-k+\frac{1}{2}w)^2-m^2+i\epsilon]}, \quad \text{(B10)}$$

where

$$A_1=[m_0-E(k)][(a+2b\alpha(q,k))Q_l^2(\alpha(q,k))$$
$$-2b\delta_{l0}Q_l(\alpha(q,k))$$
$$+c\{-l^{-1}\delta_{l0}+Q_{l-1}^2(\alpha(q,k))l(2l+1)^{-1}$$
$$+Q_{l+1}^2(\alpha(q,k))(l+1)(2l+1)^{-1}\}], \quad \text{(B11)}$$

$$A_2=2|\mathbf{q}||\mathbf{k}|[E(q)+m]^{-1}[(a+2b\alpha(q,k))Q_{l-1}(\alpha(q,k))$$
$$\times Q_l(\alpha(q,k))-b\delta_{l1}Q_l(\alpha(q,k))+c\{(2l+1)^{-1}$$
$$\times(2l-1)^{-1}Q_l(\alpha(q,k))Q_{l-1}(\alpha(q,k))+(l+1)(2l+1)^{-1}$$
$$\times Q_l(\alpha(q,k))Q_{l+1}(\alpha(q,k))+(l-1)(2l-1)^{-1}$$
$$\times Q_{l-2}(\alpha(q,k))Q_{l-1}(\alpha(q,k))\}], \quad \text{(B12)}$$

$$A_3=-\mathbf{q}^2[k_0+E(k)][E(q)+m_0]^{-2}[(a+2b\alpha(q,k))$$
$$\times Q_{l-1}^2(\alpha(q,k))-2b\delta_{l1}Q_{l-1}(\alpha(q,k))+c\{-(l-1)^{-1}\delta_{l1}$$
$$+Q_{l-2}^2(\alpha(q,k))(l-1)(2l-1)^{-1}+Q_l^2(\alpha(q,k))$$
$$\times l(2l-1)^{-1}\}]. \quad \text{(B13)}$$

It should be noted that the fictitious mass (B3) does not appear explicitly in the result. The singularity at $l=1$ comes from the Q_{l-2}^2 in the last term of (B13) and this leads to Eq. (1) of the text.